Gaussain Fits to the PAPER Antenna Beam Pattern
Katie Peek, May 2005

My task was to characterize the beam pattern of the PAPER antenna across the sky (angle from zenith θ, azimuth ϕ) and across frequency ν. The data I fit came in the form of antenna response values at every degree across the sky for 150, 160, 170, 180, 190, and 200 MHz. The data files came from Rich Bradley's simulations, and the versions I used (from 4 May 2005) are in `~kpeek/eor/gains/Beampatts/`. I did some preliminary examinations of the data and determined that a given slice in azimuth could be well characterized by a Gaussian. I set out to fit Gaussians to the beam pattern across azimuth and frequency to develop a functional form of the beam pattern that could allow the antenna gain to be calculated at any (θ, ϕ, ν).

Step I: Fitting Gaussians to the Beam Pattern.
First, I wrote a program to fit a Gaussian to the beam pattern at every azimuth, with increments of half a degree. The IDL programs `patterning`, `cut`, and `beamy`, contained in `~kpeek/eor/gains/beampatt.pro`[1], together perform the Gaussian fitting. The fit, actually executed with IDL's built-in `gauss_fit` routine, is characterized according to the standard Gaussian equation:

$$P(x) = A_0 \cdot e^{-z^2/2}, \tag{1}$$

where

$$z \equiv \frac{x - A_1}{A_2}. \tag{2}$$

In other words, A_0 represents the amplitude of the Gaussian, A_1 is its offset in x, and A_2 is its standard deviation σ. (I shall continue use x to mean θ.) A sample Gaussian fit is included in Figure 1. The fits were generally quite good, with small residuals. Fractional residuals are plotted in Figure 2.

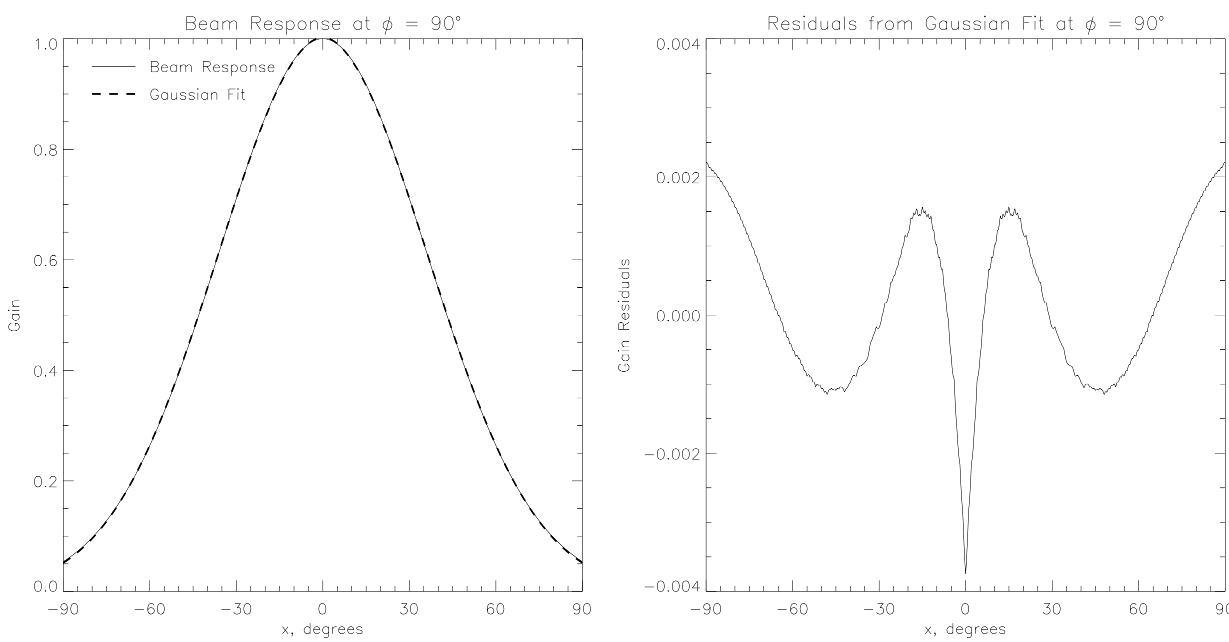

Figure 1: Sample Gaussian fit to beam response at an azimuthal angle of $\phi = 90^o$ and $\nu = 180MHz$. This particular fit is characterized by amplitude $A_0 = 1.00416$, x-offset $A_1 = -1.08245^o$, and standard deviation $A_2 = 37.3987^o$.

[1] All files are henceforth assumed to be in the directory `~kpeek/eor/gains/`.

Step II: Fitting Gaussian Coefficients as a Function of Azimuth.
Once the Gaussian was characterized at each azimuth, I worked to find a functional form to gracefully represent its coefficients as a function of angle ϕ. I had three parameters to fit: A_0, A_1, and A_2. The best fit came from the function

$$A_0(\phi) = \frac{1}{2}B_0 + B_1 \cos(2\phi) + B_2 \cos(4\phi) + \ldots + B_5 \cos(10\phi). \tag{3}$$

Since I had calculated Gaussians at every half-degree for a total of 720 points, fitting a six-term cosine function was reasonable. In my initial fit, I had used every cosine term ($\cos\phi$ and $\cos(3\phi)$, for example), but I found the coefficients for the odd-numbered terms were very small and uncertain, so using even-numbered terms only provided a better and more efficient fit. The B-coefficients were calculated by the Fourier formula

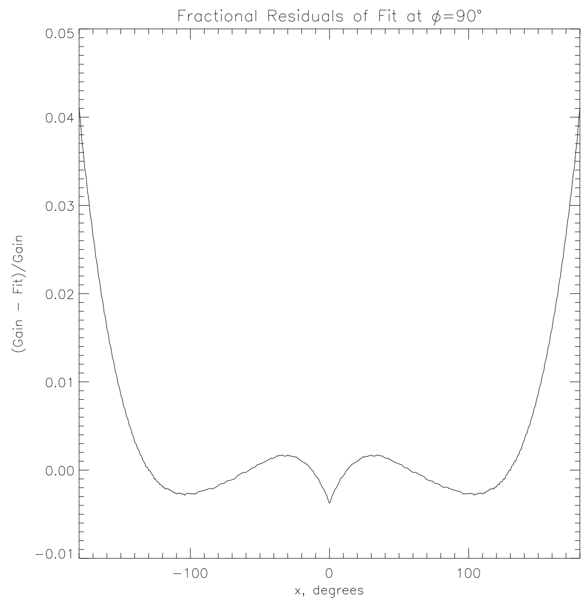

Figure 2: Fractional residuals for beam response fit in Figure 1

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx, \tag{4}$$

where $f(x)$ is the function to be fit. I performed an identical fit to all three A-coefficients at each frequency. The IDL programs I wrote to do the fitting, `polyfit` and `cosfit`, are both contained in the file `polyfit.pro`. The fits and residuals are plotted in Figures 4–9, at the end of this report. The values of the calculated B-coefficients are listed in Tables 1–3, and also in the files `writeup/Bs.*.dat`.

Step III: Fitting Cosine Fit Coefficients as a Function of Frequency.
The next step was to extrapolate between the ten-MHz increments at which I had done the Gaussian and cosine fitting. I examined each of the B-coefficients as a function of frequency (six data points) and determined that a cubic polynomial would be an effective way to extend my functional characterization. I chose a function of the form

$$B_0(\nu) = C_0 + C_1\nu + C_2\nu^2 + C_3\nu^3. \tag{5}$$

In this case I was fitting four terms to six data points, which means it wasn't hard to get my curves to look pretty good. But since I knew very little about how the beam pattern changed between the 10-MHz anchors, it seemed the most reasonable thing to do. I performed the fit described in equation 5 for each B-coefficient and each A-parameter. The IDL programs I wrote for the task, `polyfit` and `cubefit`, are in the file `polyfit.pro`. The meta-fits appear as Figures 10–12 at the end of this report. Note that the lower-quality fits correspond to smaller B-values, making them less important. The values of the C-coefficients are listed in Tables 4–6, and are also available in `writeup/Cs.*.dat`.

Step IV: Testing the Results
I haven't performed a robust test on my results; that will come when I write the program to calculate the beam pattern at a given position and frequency. I did try creating Gaussians at a few different positions and frequencies. For the example included in Figure 3, I picked a frequency of 175 MHz and calculated the B-coefficients for amplitude, x-offset, and standard deviation using equation 5 and the values Tables 4–6. Once I had the B-coefficients at my desired frequency, I calculated the A-parameters of the Gaussian according to equation 3 and the values in Tables 1–3. With A_0, A_1, and A_2 in hand, I was able to create a Gaussian at $\phi = +90^o$ and $\nu = 175$ MHz. The results are plotted in Figure 3, and they agree reasonably well with the 170 and 180 MHz plots in Figures 4–9.

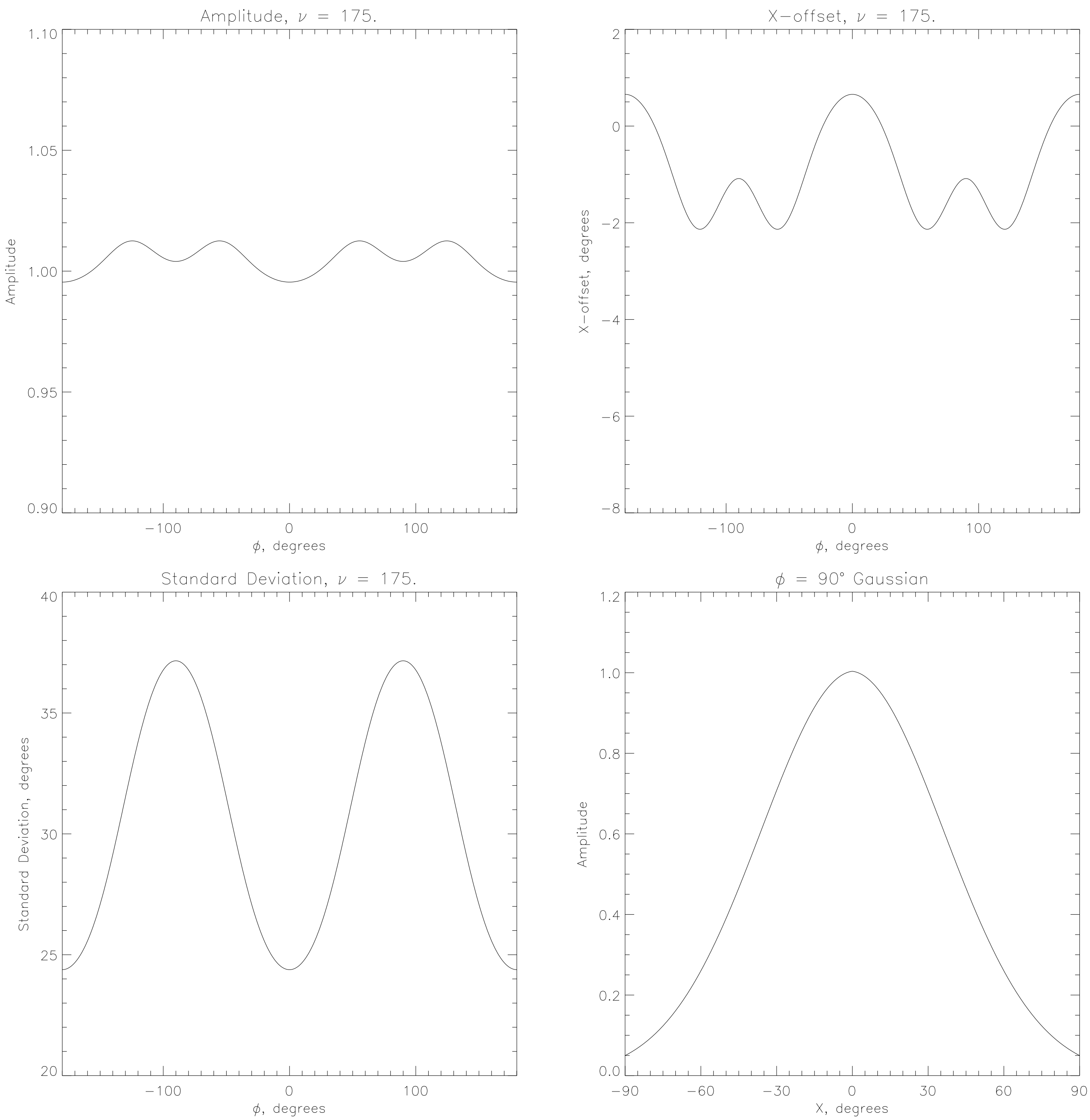

Figure 3: Sample Gaussian created with coefficients listed in Tables 1–6. Top left is the Gaussian amplitude at 175 MHz as a function of azimuthal angle ϕ; top right is the X-offset at 175 MHz as a function of ϕ; bottom left is the standard deviation at 175 MHz as a function of ϕ; bottom right is the Gaussian at $\phi = 90^o$ described by the previous three parameters.

Postscript: A Word About Coordinates.
In the code to calculate the beam response, I use several different coordinate systems. Here's a brief explanation of each:

Coord.	Explanation
(h, d)	Hour angle h and declination d.
(x, y, z)	Cartesian coordinates with x in the NCP direction, y pointing toward the west point, and z toward the celestial equator at $h = 0$.
(x', y', z')	Cartesian; x' toward north point, y' toward west point ($y' = y$), and z' toward zenith. The (x', y', z') system is (x, y, z) rotated by latitude angle.
(α, ξ)	Spherical coordinates with α as altitude (zero at horizon) and ξ as azimuth (zero at north point).
(θ, ϕ)	Spherical coordinates in the antenna frame; θ is measured down from zenith ($\theta = \pi/2 - \alpha$) and ϕ accounts for antenna rotation relative to celestial coordinates ψ such that $\phi = \xi - \psi$.

Table 1: Coefficients for Cosine Fit to Standard Deviation as a Function of ϕ.
$A_2(\phi) = B_0 + B_1 \cos(2\phi) + B_2 \cos(4\phi) + B_3 \cos(6\phi) + B_4 \cos(8\phi) + B_5 \cos(10\phi)$

$A_2(\phi)$	B_0	B_1	B_2	B_3	B_4	B_5
150 MHz	64.319656	-5.5676199	-1.8148823	0.33104367	0.15102909	-0.014553658
160 MHz	62.741876	-5.9611397	-0.87397702	0.25559184	0.066580114	-0.012865680
170 MHz	61.503490	-6.3788426	-0.00879202	0.14335534	-0.000251318	-0.003815715
180 MHz	60.745233	-6.6135390	0.49542050	0.05454166	-0.023359798	0.005630303
190 MHz	60.393574	-6.6302313	0.73597069	-0.02094681	-0.023844217	0.010312195
200 MHz	60.216282	-6.4889824	0.97551269	-0.13459642	-0.012474784	0.014129489

Table 2: Coefficients for Cosine Fit to X-offset as a Function of ϕ.
$A_1(\phi) = B_0 + B_1 \cos(2\phi) + B_2 \cos(4\phi) + B_3 \cos(6\phi) + B_4 \cos(8\phi) + B_5 \cos(10\phi)$

$A_1(\phi)$	B_0	B_1	B_2	B_3	B_4	B_5
150 MHz	-6.2922256	1.5375599	2.7382597	-0.42653821	-0.26926847	0.031726275
160 MHz	-4.1757405	1.2413700	1.7869122	-0.38772117	-0.11660724	0.027160251
170 MHz	-2.3682211	1.1349107	0.92773259	-0.30793852	0.004792800	0.010697457
180 MHz	-1.2451792	1.1722832	0.42265290	-0.23350408	0.050755542	-0.005333881
190 MHz	-0.0822924	1.1256321	0.18328895	-0.16012693	0.059171480	-0.013951783
200 MHz	1.6516188	1.0074870	-0.02554527	-0.05636530	0.05616263	-0.021953365

Table 3: Coefficients for Cosine Fit to Amplitude as a Function of ϕ.
$A_0(\phi) = B_0 + B_1 \cos(2\phi) + B_2 \cos(4\phi) + B_3 \cos(6\phi) + B_4 \cos(8\phi) + B_5 \cos(10\phi)$

$A_0(\phi)$	B_0	B_1	B_2	B_3	B_4	B_5
150 MHz	2.0405854	-0.007673359	-0.020625450	0.002011823	0.002999090	-0.0001914999
160 MHz	2.0255133	-0.005978963	-0.013254365	0.001928361	0.001480007	-0.0002341947
170 MHz	2.0132516	-0.005642760	-0.007000103	0.001683671	0.000318197	-0.0001421330
180 MHz	2.0065768	-0.006010287	-0.003634081	0.001408656	-0.000144212	-3.748087e-05
190 MHz	2.0001674	-0.005889695	-0.002047409	0.001023339	-0.000268806	3.085017e-05
200 MHz	1.9922913	-0.005383045	-0.000833252	0.000495343	-0.000303539	0.0001020922

Table 4: Coefficients for Cubic Fit to Standard Deviation Coefficients as a Function of ν.

$$f(\nu) = C_0 + C_1 \cdot \nu + C_2 \cdot \nu^2 + C_3 \cdot \nu^3$$

$f(\nu)$	C_0	C_1	C_2	C_3
B_0	183.99317	-1.6249565	0.0069672259	-9.6826907e-06
B_1	-29.244673	0.57229158	-0.0041730996	9.4038248e-06
B_2	-74.733475	0.99345539	-0.0043074107	6.1619539e-06
B_3	3.3041833	-0.037977133	0.00017285586	-3.4421417e-07
B_4	9.1105618	-0.12763799	0.00058072483	-8.5291801e-07
B_5	2.6146875	-0.046834018	0.00027398547	-5.2419876e-07

Table 5: Coefficients for Cubic Fit to X-offset Coefficients as a Function of ν.

$$f(\nu) = C_0 + C_1 \cdot \nu + C_2 \cdot \nu^2 + C_3 \cdot \nu^3$$

$f(\nu)$	C_0	C_1	C_2	C_3
B_0	-373.77001	5.9933023	-0.032753724	6.0860359e-05
B_1	105.90908	-1.7623625	0.0098737879	-1.8423046e-05
B_2	72.094475	-0.92672015	0.0038927716	-5.3087386e-06
B_3	2.0078893	-0.045110799	0.00024801570	-3.7067584e-07
B_4	-18.188548	0.26325797	-0.0012543987	1.9708660e-06
B_5	-3.7902866	0.068599528	-0.00040342212	7.7325363e-07

Table 6: Coefficients for Cubic Fit to Amplitude Coefficients as a Function of ν.

$$f(\nu) = C_0 + C_1 \cdot \nu + C_2 \cdot \nu^2 + C_3 \cdot \nu^3$$

$f(\nu)$	C_0	C_1	C_2	C_3
B_0	4.3197202	-0.036236564	0.00019216586	-3.4582946e-07
B_1	-0.64876376	0.010860822	-6.0990572e-05	1.1385919e-07
B_2	-0.68774619	0.0094422778	-4.3093392e-05	6.5261223e-08
B_3	-0.0011915388	1.9346063e-05	2.1812495e-07	-1.3628154e-09
B_4	0.19266323	-0.0028355537	1.3838885e-05	-2.2424197e-08
B_5	0.040644458	-0.00070476657	4.0026507e-06	-7.4626130e-09

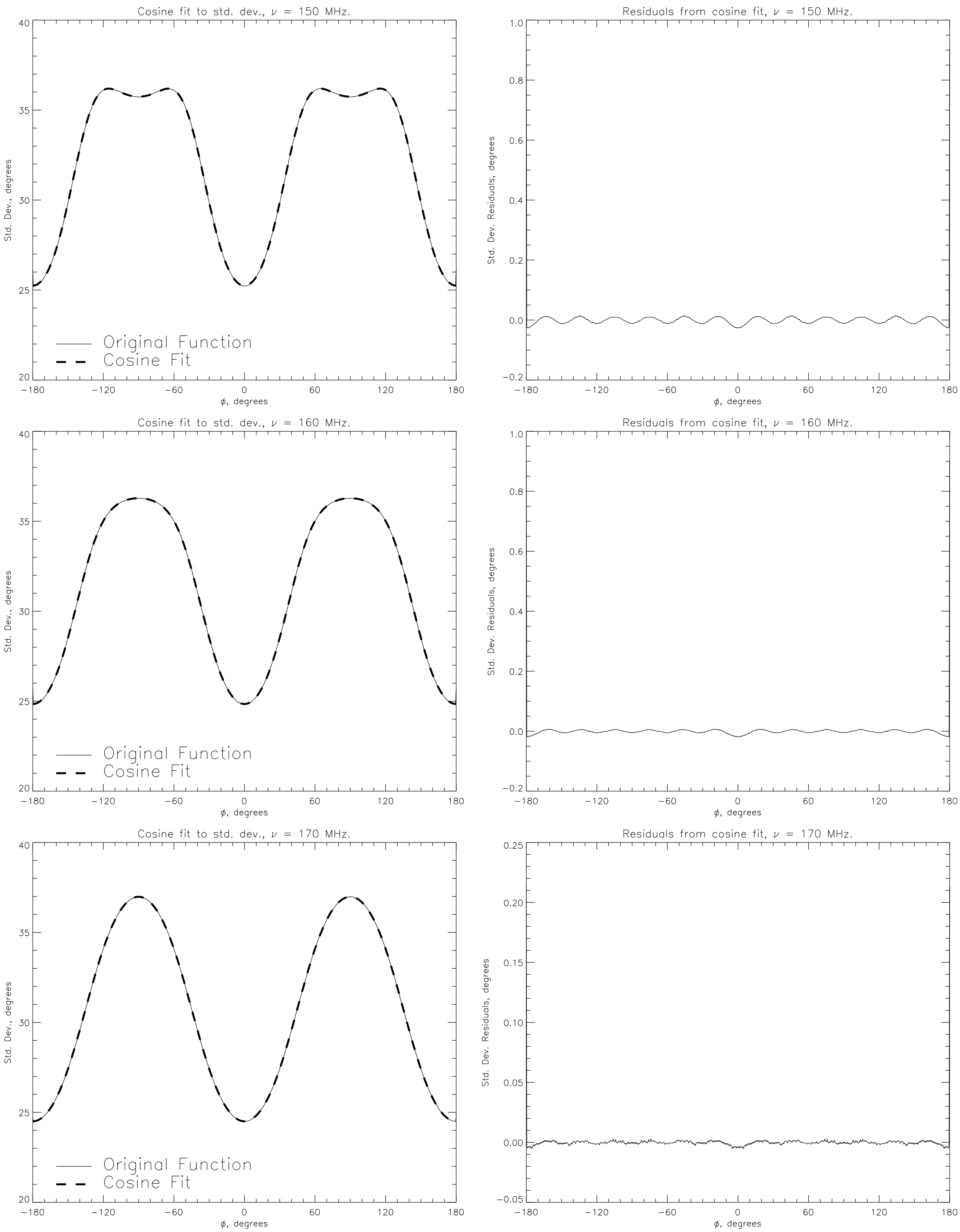

Figure 4: Cosine Fits to Standard Deviation (A_2) as a function of ϕ, with residuals, 150 MHz–170 MHz.

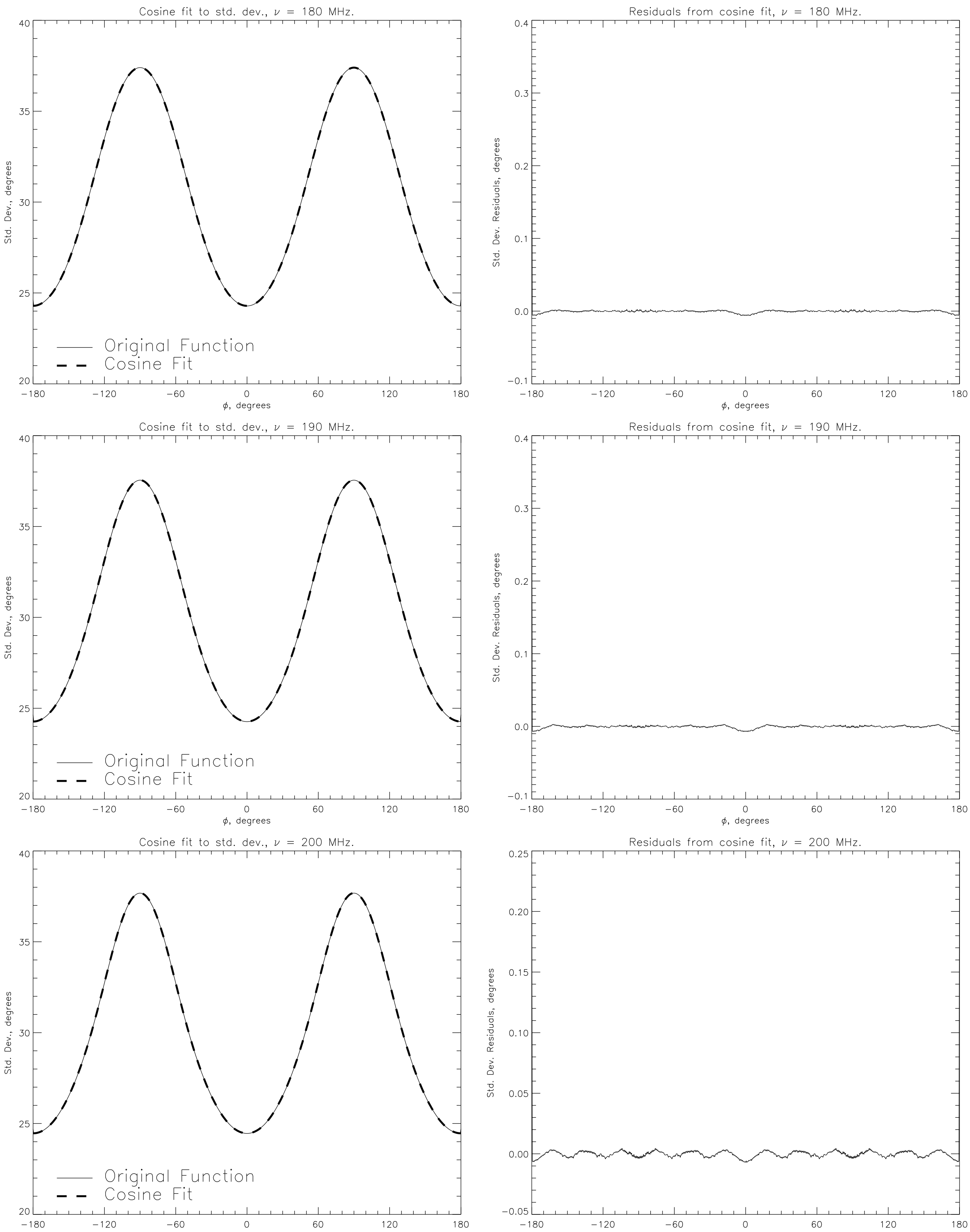

Figure 5: Cosine Fits to Standard Deviation (A_2) as a function of ϕ, with residuals, 180 MHz–200 MHz.

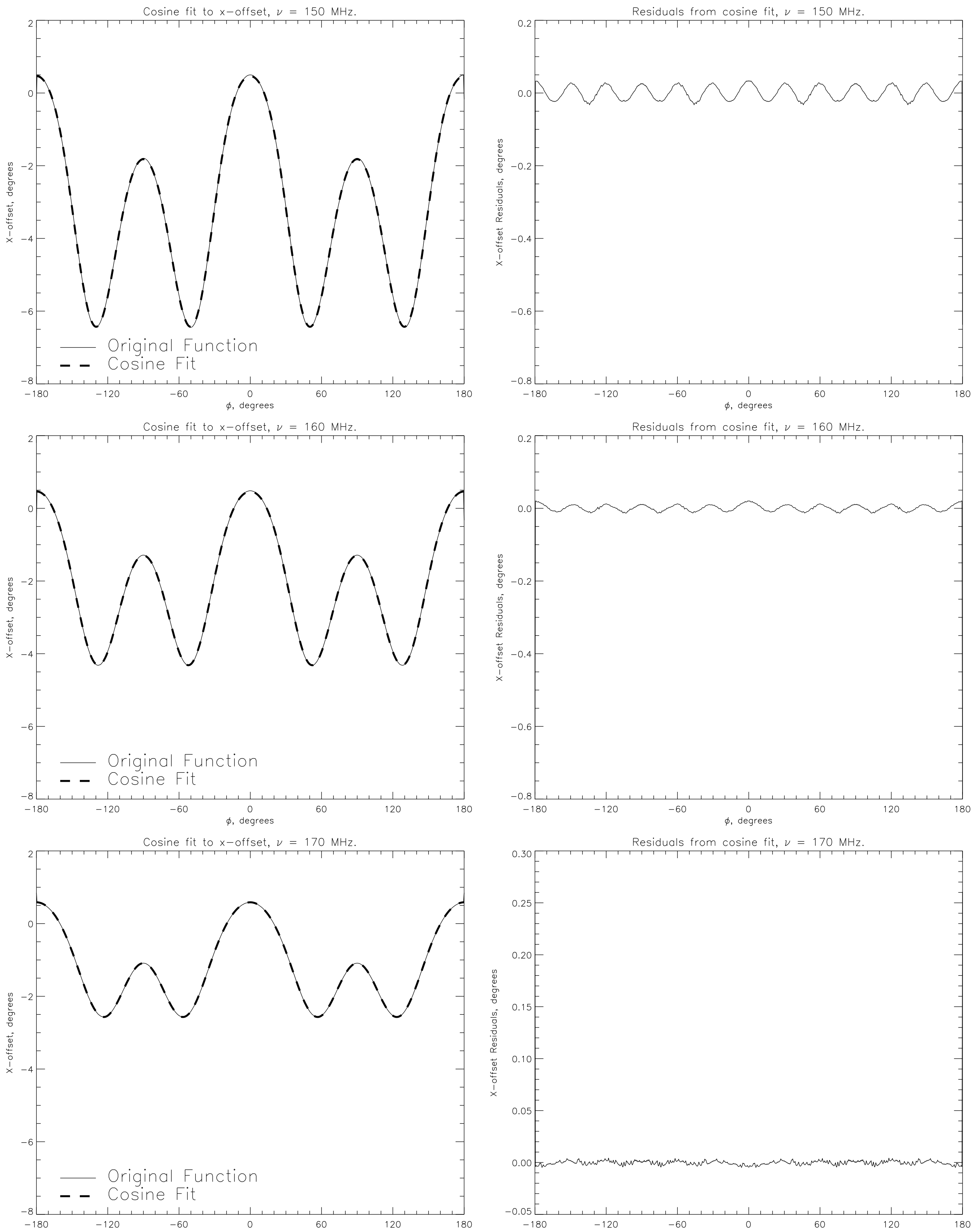

Figure 6: Cosine Fits to X-offset (A_1) as a function of ϕ, with residuals, 150 MHz–170 MHz.

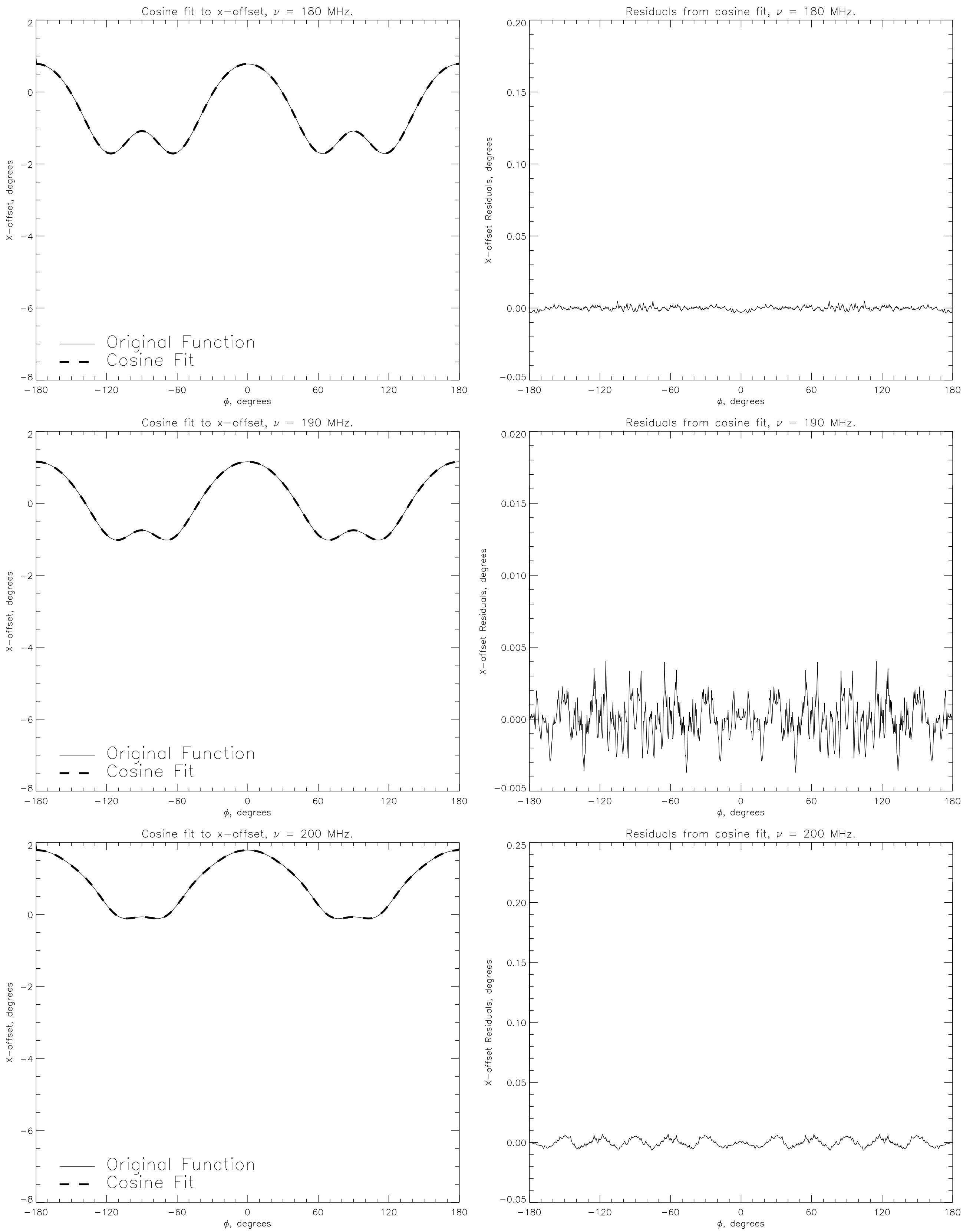

Figure 7: Cosine Fits to X-offset (A_1) as a function of ϕ, with residuals, 180 MHz–200 MHz.

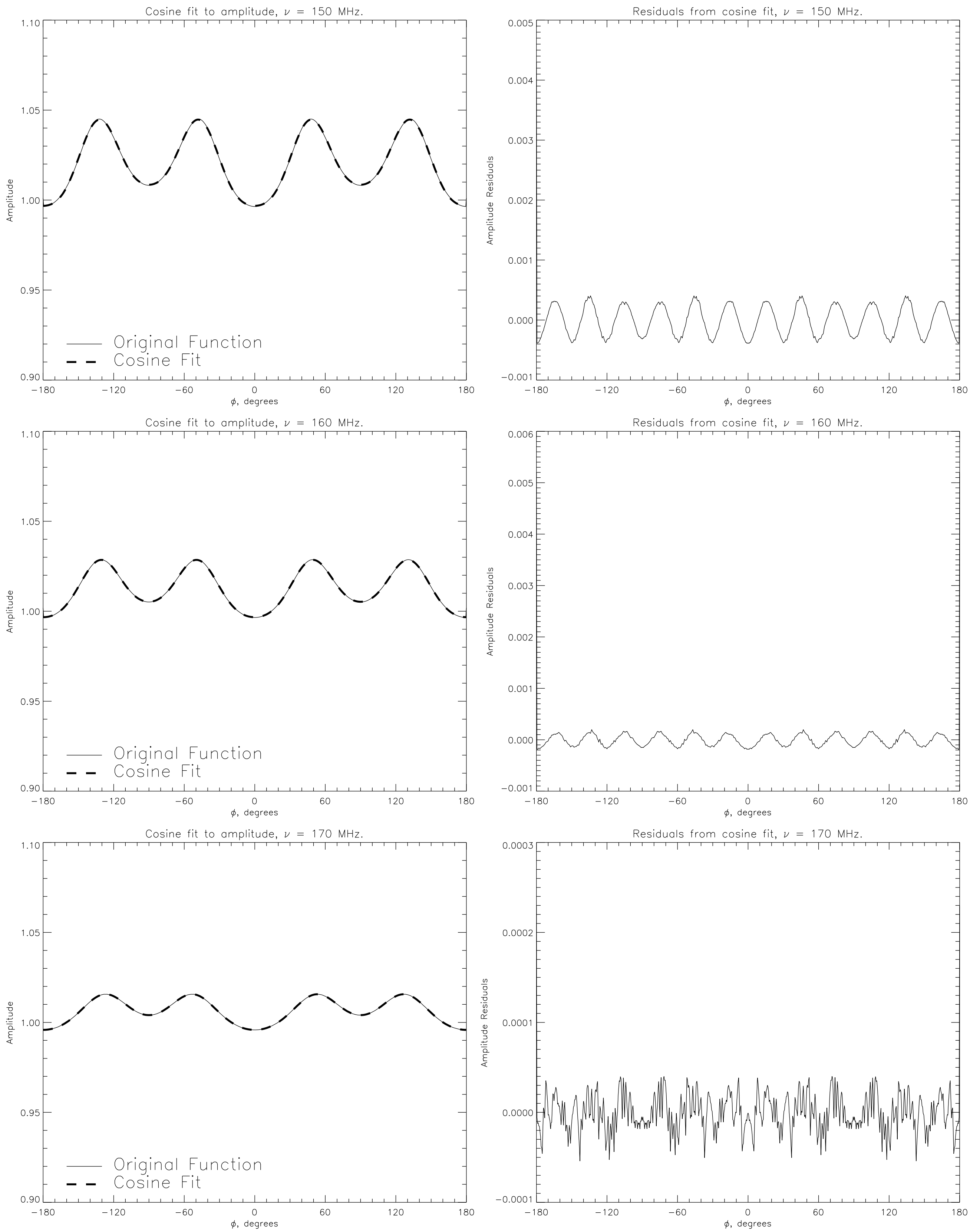

Figure 8: Cosine Fits to Amplitude (A_0) as a function of ϕ, with residuals, 150 MHz–170 MHz.

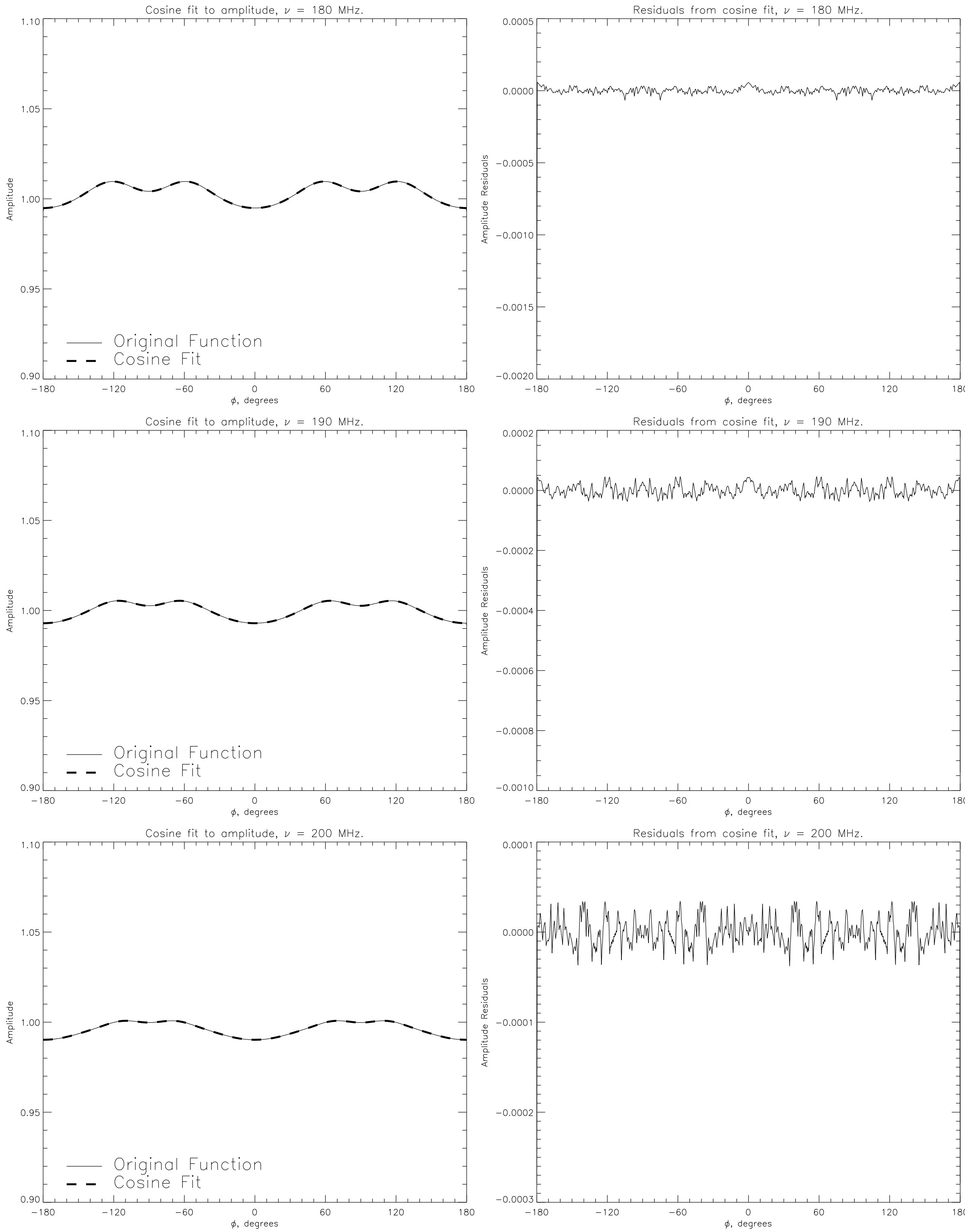

Figure 9: Cosine Fits to Amplitude (A_0) as a function of ϕ, with residuals, 180 MHz–200 MHz.

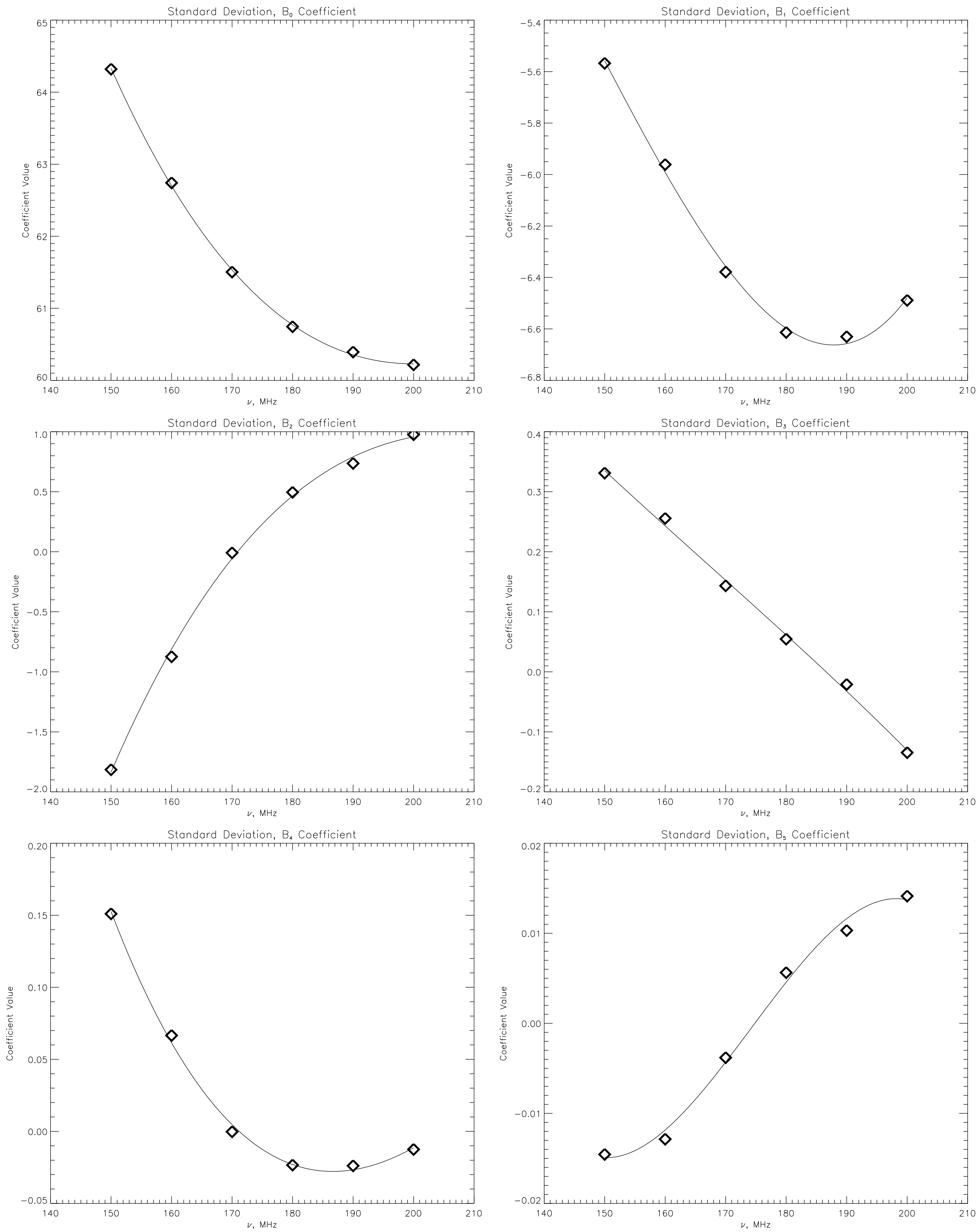

Figure 10: Fits to each B-coefficient for the standard deviation A_2 as a function of frequency ν.

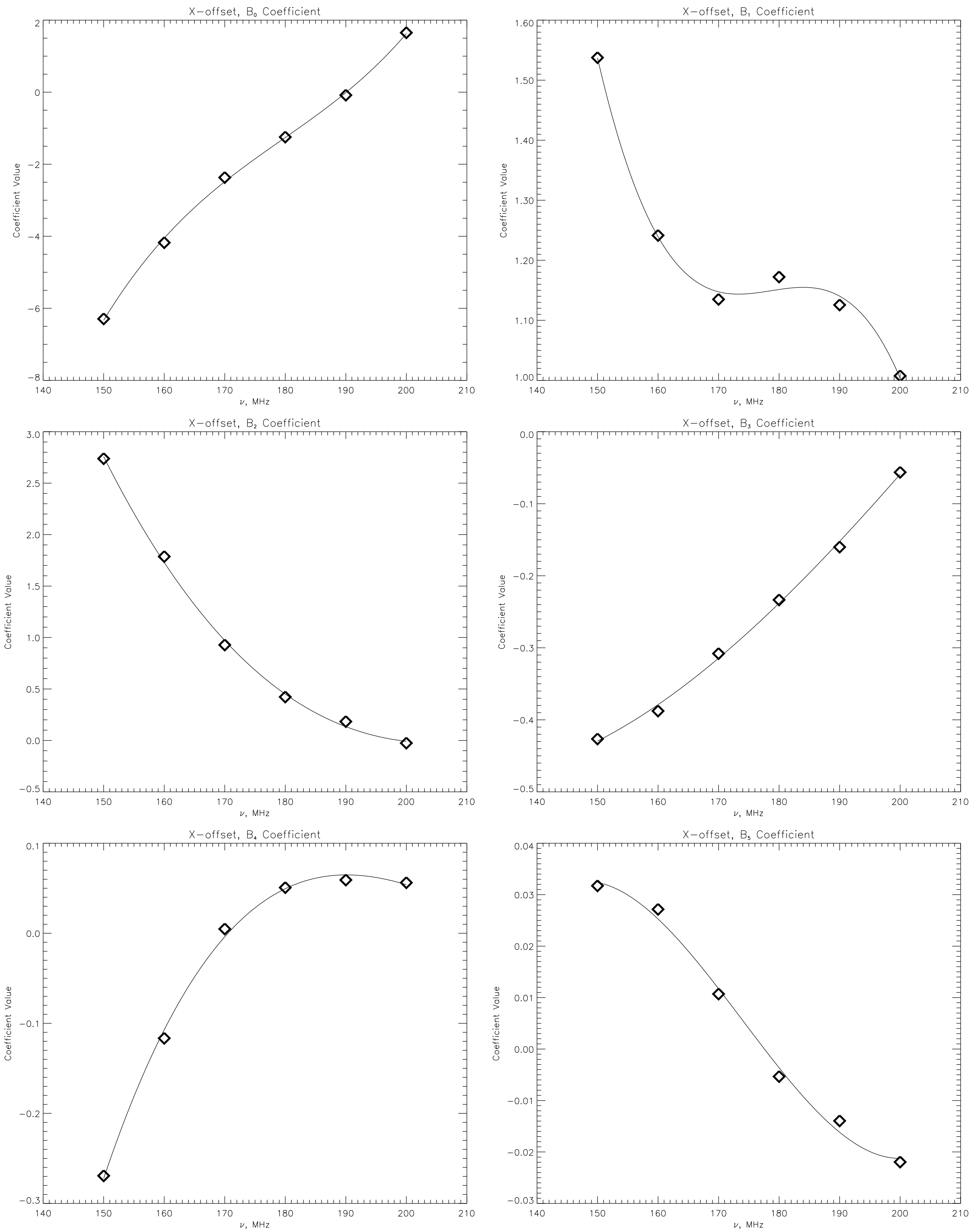

Figure 11: Fits to each B-coefficient for the x-offset A_1 as a function of frequency ν.

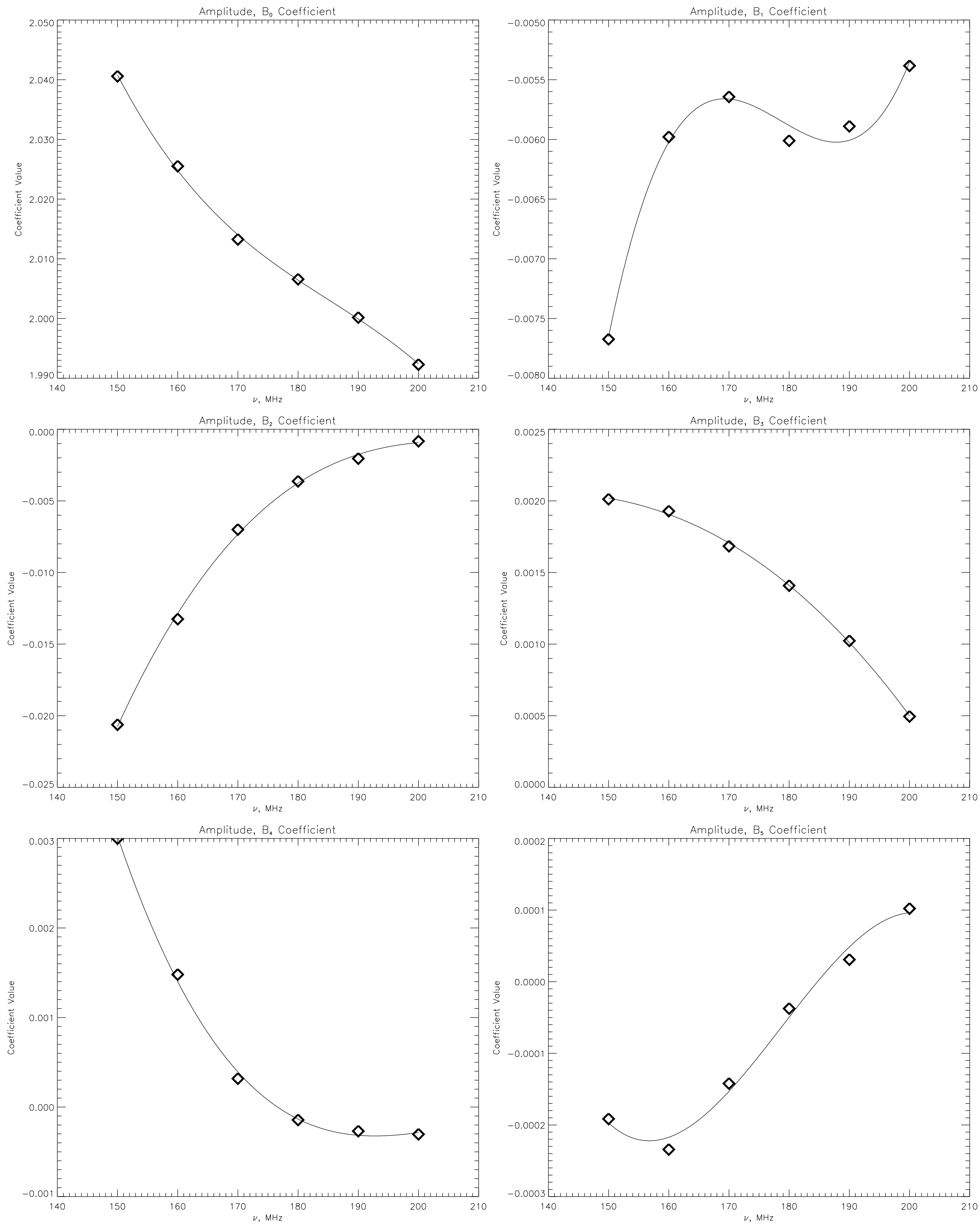

Figure 12: Fits to each B-coefficient for the amplitude A_0 as a function of frequency ν.